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ON THE INVERSE PROBLEM OF THE CALCULUS OF VARIATIONS *

BY G. A. BLISS

IN the second volume of the ANNALS,† Prof. Osgood has shown in a very interesting way why an arc

$$y = y(x), \qquad x_0 \leqq x \leqq x_1,$$

which gives a minimum (or a maximum) value to an integral of the form

$$\int g(x, y, y_x)\,dx, \tag{1}$$

must be an extremal, that is, a solution of the differential equation

$$g_y - \frac{d}{dx} g_{y'} = 0. \tag{2}$$

This equation is of the second order, and its solutions are therefore a two-parameter family of curves.

In the so-called inverse problem of the calculus of variations the two-parameter family of solutions of equation (2) is given in advance, and it is required then to find the integrals of the form (1) for which these curves are the extremals. In connection with some problems in differential geometry Darboux‡ has found that there is an infinity of such integrals, and showed how they may be determined. For more general cases the corresponding problems are discussed in an interesting article by Hirsch.§

In the present paper it is proposed to apply the method of Darboux to integrals in the form

$$\int f(x, y, \tau)\,\sqrt{x_t^2 + y_t^2}\,dt. \tag{3}$$

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* The first part of this paper, §§1, 2, was presented before The Chicago Section of the American Mathematical Society, April 22, 1905, under the title "The inverse problem of the calculus of variations in parametric representation."

† Ser. 2, vol. 2, p. 107; 1901.

‡ *Leçons sur la théorie générale des surfaces*, vol. 3, §§604–605.

§ *Mathematische Annalen*, vol. 49 (1897), p. 49.

The arcs along which the integral is taken are supposed to be represented in the form

$$(4)\qquad x = x(t), \quad y = y(t), \qquad t_0 \leqq t \leqq t_1,$$

and τ is the angle between the tangent and the x-axis defined by the equations

$$\cos\tau = \frac{x_t}{\sqrt{x_t^2 + y_t^2}}, \quad \sin\tau = \frac{y_t}{\sqrt{x_t^2 + y_t^2}},$$

or, what is the same thing, by

$$(5)\qquad \tau = \tau_0 + \int_{t_0}^{t} \frac{x_t y_{tt} - x_{tt} y_t}{x_t^2 + y_t^2}\,dt,$$

where τ_0 is the initial value of τ at the point $t = t_0$. The integral (3) evidently reduces to (1) when

$$f(x, y, \tau) = g(x, y, \tan\tau)\cos\tau.$$

The geometrical advantage in using an integral of the form (3) instead of (1) is due to the fact that the arcs over which it is taken may go in any direction, while in the form in which Darboux considered the problem difficulties are always encountered when the tangent to the curve becomes parallel to the y-axis, that is, when y_x becomes infinite. For the inverse problem the integral (3) is also decidedly more convenient than the well-known integral

$$(6)\qquad \int_{t_0}^{t_1} F(x, y, x_t, y_t)\,dt$$

which is usually used when the curves are taken in parametric representation. This is because the function f is determined by integrating a single linear partial differential equation instead of the system of such equations which present themselves when the solution of the problem for the integral (6) is attempted.

The first two sections below are devoted to the derivation for the integral (3) of the so-called Euler equation corresponding to equation (2), and to the determination of the integrals in the form (3) which have a given family of extremals. The integrals which Darboux gives and those which are found in §2 involve two arbitrary functions. In §3 it is shown what further conditions, besides the prescription of the extremals, suffice to determine the integral uniquely, and in §4 the results of the preceding sections are applied to the determination of the integrals which are minimized by straight lines.

1. Euler's equation. The derivation of Euler's equation for the extremals of the integral (3) has been made in a previous paper,* but for the sake of completeness it will be repeated here. An arc E in the form (4) is supposed to give to the integral a smaller value than that given by any other curve in its neighborhood with the same endpoints. A variation V of E in the form

$$(V) \qquad X = x(t) + a\xi(t), \qquad Y = y(t) + a\eta(t), \qquad t_0 \leqq t \leqq t_1,$$

where ξ and η are two functions of t vanishing at t_0 and t_1, can be made to lie as near as is desired to E by taking the parameter a sufficiently small, and for $a = 0$ the arc V coincides with E itself. It is evident then that the integral

$$I(a) = \int_{t_0}^{t_1} f(X, Y, T)\sqrt{X_t^2 + Y_t^2}\, dt$$

is a function of a which must have a minimum, and consequently the derivative $\frac{dI}{da}$ must vanish, for $a = 0$.

The derivative can be readily calculated when it is noticed that

$$X_a = \xi, \quad X_{ta} = \xi_t, \quad Y_a = \eta, \quad Y_{ta} = \eta_t,$$

and that

$$T_a = \frac{X_t\eta - Y_t\xi}{X_t^2 + Y_t^2}$$

since

$$\cos T = \frac{X_t}{\sqrt{X_t^2 + Y_t^2}}, \quad \sin T = \frac{Y_t}{\sqrt{X_t^2 + Y_t^2}}.$$

Then

$$\left[\frac{dI}{da}\right]_{a=0} = \int_{t_0}^{t_1} \left\{ f_x\xi + f_y\eta + (f\cos\tau - f_\tau\sin\tau)\xi_s + (f\sin\tau + f_\tau\cos\tau)\eta_s \right\} ds,$$

where

$$ds = \sqrt{x_t^2 + y_t^2}\, dt,$$

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* A generalization of the notion of angle, *Transactions of the American Mathematical Society*, vol. 7 (1906), p. 185.

or by a partial integration

$$(7)\qquad \left[\frac{dI}{da}\right]_{a=0}=\int_{t_0}^{t_1}\left\{\left[f_x-\frac{d}{ds}(f\cos\tau-f_\tau\sin\tau)\right]\xi\right.$$
$$\left.+\left[f_y-\frac{d}{ds}(f\sin\tau+f_\tau\cos\tau)\right]\eta\right\}ds.$$

This expression must be zero however ξ and η are chosen, and by the usual argument in the calculus of variations it follows that along the arc E the two expressions

$$P=f_x-\frac{d}{ds}(f\cos\tau-f_\tau\sin\tau),\qquad Q=f_y-\frac{d}{ds}(f\sin\tau+f_\tau\cos\tau),$$

must vanish. These conditions upon E are not independent, for the relation

$$(8)\qquad P\cos\tau+Q\sin\tau=0$$

holds between P and Q, and one verifies readily that the vanishing of the two expressions P and Q is equivalent to the vanishing of the expression T defined by the equation

$$(9)\qquad P\sin\tau-Q\cos\tau=T.$$

In fact from (8) and (9)

$$P=T\sin\tau,\quad Q=-T\cos\tau.$$

From either of these equations the value of T is easily calculated.

Along any arc E which minimizes the integral (3) *the expression*

$$(10)\qquad T(x,y,\tau,\tau_s)=f_x\sin\tau-f_y\cos\tau+f_{x\tau}\cos\tau+f_{y\tau}\sin\tau+(f+f_{\tau\tau})\tau_s$$

must vanish identically. The equation

$$(11)\qquad T(x,y,\tau,\tau_s)=0$$

is called the Euler equation for the integral (3).

2. Determination of the integrals which have given extremals. Consider a family of curves

$$(12)\qquad x=x(t,u,v),\quad y=y(t,u,v),$$

involving the two parameters u and v. From equation (5) the direction angle τ is also a function of t, u, v,

$$\tau = \tau(t, u, v), \tag{13}$$

and it will be supposed that the three equations (12) and (13) have single-valued solutions

$$t = t(x, y, \tau), \quad u = u(x, y, \tau), \quad v = v(x, y, \tau) \tag{14}$$

for all points (x, y) in a certain region R of the xy-plane, and for arbitrary values of τ. In other words through each point of R there passes one and but one of the curves (12) in a given direction τ. All six of the functions are supposed to be continuous and have continuous first derivatives.

A family of curves of this sort will consist of solutions of a system of differential equations of the form

$$x_s = \cos\tau, \quad y_s = \sin\tau, \quad \tau_s = \phi(x, y, \tau). \tag{15}$$

For from equation (13) and the expression for s_t, the curvature $\tau_s = \frac{\tau_t}{s_t}$ is seen to be a function of t, u, v, and by equations (14) of x, y, τ. Conversely, when the three equations (15) are given, it follows from the well-known theory of differential equations that there is but one two-parameter family of curves which satisfy them. The assumption that the curves (12) are the extremals of the integral (3), and the assumption that the extremals must satisfy a given system of differential equations in the form (15), are therefore equivalent.

If the extremals of the integral are to be the solutions of the system (15), then from (11) the function f must satisfy the equation

$$f_x \sin\tau - f_y \cos\tau + f_{x\tau}\cos\tau + f_{y\tau}\sin\tau + (f + f_{\tau\tau})\phi = 0 \tag{16}$$

identically in x, y, τ. This is a partial differential equation of the second order for f and not easily solvable by well-known methods. But if it is differentiated partially for τ, another differential equation is found,

$$M_x \cos\tau + M_y \sin\tau + M_\tau \phi + M\phi_\tau = 0, \tag{17}$$

which is linear in the function

$$M = f + f_{\tau\tau} \tag{18}$$

and its derivatives of the first order.

The equations (17) and (18) can now be solved by a method considerably simpler than that of Darboux since it avoids the use of the theory of partial differential equations. By means of equations (12) and (13), $M(x, y, \tau)$ may be thought of as a function of t, u, v. From equation (17), then,

$$\frac{dM}{dt} + M\phi_\tau\sqrt{x_t^2 + y_t^2} = 0,$$

since $\tau_s = \phi$ and $M_t = M_s\sqrt{x_t^2 + y_t^2}$. Therefore

$$M = G(u, v)e^{-\int\phi_\tau\sqrt{x_t^2 + y_t^2}\,dt}, \tag{19}$$

where x, y, τ in ϕ_τ are supposed to be replaced by their values in terms of t, u, v, and the integral in the exponent stands for some particular function of t, u, v whose derivative for t is $\phi_\tau\sqrt{x_t^2 + y_t^2}$. The function $G(u, v)$ is an arbitrary function of u and v. The determination of M as a function of t, u, v in this manner means that it is also determined as a function of x, y, τ on account of equations (14).

It follows of course from the foregoing discussion that the function $M(x, y, \tau)$ defined by equation (19) satisfies the differential equation (17), but this can also be readily verified by direct substitution. The first member of (17) becomes simply

$$\begin{aligned} &M_u(u_x\cos\tau + u_y\sin\tau + u_\tau\phi) \\ &+ M_v(v_x\cos\tau + v_y\sin\tau + v_\tau\phi) \\ &+ M_t(t_x\cos\tau + t_y\sin\tau + t_\tau\phi) + M\phi_\tau, \end{aligned}$$

which vanishes since u and v are constant along an extremal and

$$M_t(t_x\cos\tau + t_y\sin\tau + t_\tau\phi) = M_t\frac{dt}{ds} = -M\phi_\tau,$$

from equation (19).

It should perhaps be mentioned that the functions u and v above are not the only ones which might be used in the determination of M. If u and v are any two independent solutions of the abbreviated equation

$$\Phi_x\cos\tau + \Phi_y\sin\tau + \Phi_\tau\phi = 0, \tag{20}$$

and if ψ is a particular solution of (17), then it is true that $\dfrac{M}{\psi}$, u, v, thought of as functions of the four variables x, y, τ, M, are independent solutions of the auxiliary linear differential equation

$$\Phi_x \cos\tau + \Phi_y \sin\tau + \Phi_\tau \phi - \Phi_M M \phi_\tau = 0.$$

From the theory of linear equations* it follows that the most general solution $M(x, y, \tau)$ of equation (17) is found by solving the equation

$$\frac{M}{\psi} = G(u, v) \tag{21}$$

for M. The functions u and v from equations (12), and the exponential factor in equation (19) are particular solutions u, v, ψ of equations (20) and (17), respectively.

It remains to derive the value of f from that of M by means of the equation (18). This is a differential equation of the second order with respect to τ, having as its most general solution†

$$f = \int_0^\tau \sin(\tau - \lambda) M(x, y, \lambda)\, d\lambda + A(x, y) \cos\tau + B(x, y) \sin\tau, \tag{22}$$

where A and B are arbitrary functions of x and y. It is not certain that the value of f just given satisfies the Euler equation (16). In fact if the expression (22) is substituted in (16), the equation

$$\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x} = \int_0^\tau (M_x \cos\lambda + M_y \sin\lambda)\, d\lambda + \phi(x, y, \tau) M(x, y, \tau) \tag{23}$$

is found. The function on the right is independent of τ since its derivative for τ is the expression (17), and consequently equation (23) may be written

$$\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x} = \phi(x, y, 0) M(x, y, 0). \tag{24}$$

If A_0 and B_0 are any two particular solutions of this last equation, then it can readily be shown that the differences $A - A_0$, $B - B_0$, in which (A, B) is any other solution, satisfy the equation

$$\frac{\partial(A - A_0)}{\partial y} = \frac{\partial(B - B_0)}{\partial x},$$

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* See for example Jordan, *Cours d'Analyse*, vol. 3 (1896), p. 314.

† For a discussion of linear differential equations with constant coefficients, see Murray, *Differential Equations*, chap. VI.

and consequently that the most general solution of (24) is

$$A = A_0 + \frac{\partial \theta}{\partial x}, \qquad B = B_0 + \frac{\partial \theta}{\partial y},$$

where θ is an arbitrary function of x and y.

Hence *the most general function* $f(x, y, \tau)$ *for which the integral* (3) *has a given set of extremals* (12), *is given by the equation*

$$(25) \quad f = \int_0^\tau \sin(\tau - \lambda)\, M(x, y, \lambda)\, d\lambda + \left(A_0 + \frac{\partial \theta}{\partial x}\right)\cos\tau + \left(B_0 + \frac{\partial \theta}{\partial y}\right)\sin\tau.$$

In this equation

$$M(x, y, \tau) = G(u, v)\, e^{-\int \phi_\tau \sqrt{x_t^2 + y_t^2}\, dt},$$

where $G(u, v)$ *is an arbitrary function of* u *and* v, *and the arguments* t, u, v *in the expression on the right can be replaced by their values* (14) *in* x, y, τ. *The pair of functions*, $A_0(x, y)$ *and* $B_0(x, y)$, *are a particular solution of the differential equation*

$$\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x} = \phi(x, y, 0)\, M(x, y, 0),$$

and θ *is an arbitrary function of* x *and* y.

3. Further conditions determining f uniquely. It is natural to ask at this point what further conditions on the function f would suffice to determine it uniquely. In order to determine the function $G(u, v)$ suppose that the region R has a simply closed curve

$$(B) \qquad x = g(\alpha), \qquad y = h(\alpha)$$

as its boundary, which nowhere satisfies the Euler differential equation (11). It is evident that no extremal can be a closed curve in R, otherwise after a complete circuit the same values of x, y, τ would correspond to different values of t, which is contrary to the hypotheses upon the functions (12). It will be supposed, furthermore, that for any fixed values of u, v corresponding to a point of R and a direction through that point, the functions (12) of t can be continued until the extremal encounters the boundary curve B. No extremal E can cross the curve B at a point of tangency with B. For two curves tangent to each other cannot cross unless they have the same curvature, and by hypothesis B can never satisfy the equation (11) which defines the curvature τ_s of E. It follows then that every extremal of the set (12) crosses the boundary curve B without being tangent to it, at one point at least.

Let now the values of $M(x, y, \tau)$ be assigned for any point (x, y) on the curve B and for all values of τ. The only restriction upon the values assigned to M, aside from the restriction that $M(a, \tau)$ as a function of a and τ shall be continuous and have continuous first derivatives, will be that the values of the quotient $\dfrac{M}{\psi}$ shall be the same at all points which the extremal E has in common with the curve B. Thus if the region R were a circle and the extremal E a straight line, $\dfrac{M}{\psi}$ would be the same for the values of x, y, τ on the straight line at both its intersection points with the circle. This restriction is necessary because from equation (21), $\dfrac{M}{\psi}$ is to be everywhere a function of u and v alone.

It follows now easily that for every value of u and v defining an extremal E in the region R, the value of $G(u, v)$ is completely determined. For the extremal E intersects B at least at one point, and there the value of M has been assigned, so that the value of G is completely determined by equation (21).

The analytical expression for $G(u, v)$ can be determined in an actual case by solving the equations

$$u = u[g(a), h(a), \tau], \qquad v = v[g(a), h(a), \tau] \tag{26}$$

for a and τ and substituting the results so found in the expression

$$\frac{M(a, \tau)}{\psi[g(a), h(a), \tau]}.$$

The values of u, v defined by equations (26) at a point b where an extremal E crosses B, are those belonging to E. The functional determinant

$$\begin{vmatrix} u_x g_a + u_y h_a & u_\tau \\ v_x g_a + v_y h_a & v_\tau \end{vmatrix} = \begin{vmatrix} u_x & u_\tau \\ v_x & v_\tau \end{vmatrix} g_a + \begin{vmatrix} u_y & u_\tau \\ v_y & v_\tau \end{vmatrix} h_a,$$

is different from zero at the intersection point. For along the extremal E

$$u_x \cos\tau + u_y \sin\tau + u_\tau \phi = 0,$$

$$v_x \cos\tau + v_y \sin\tau + v_\tau \phi = 0,$$

and therefore

$$\cos\tau : \sin\tau : \phi = \begin{vmatrix} u_y & u_\tau \\ v_y & v_\tau \end{vmatrix} : \begin{vmatrix} u_\tau & u_x \\ v_\tau & v_x \end{vmatrix} : \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix},$$

so that the functional determinant above is

$$(-\cos\tau\, g_a + \sin\tau\, h_a)\left\{ \begin{vmatrix} u_x & u_\tau \\ v_x & v_\tau \end{vmatrix}^2 + \begin{vmatrix} u_y & u_\tau \\ v_y & v_\tau \end{vmatrix}^2 \right\}.$$

This cannot vanish at a point b where E crosses B, since at such a point B and E are never tangent. Consequently from the theory of implicit functions* the solutions of the equations (26) are continuous functions of u and v with continuous first derivatives in the neighborhood of the values u, v which define any extremal E in the region R.

The only arbitrary part of f besides $G(u, v)$ is the function $\theta(x, y)$, and the values of θ can be determined by prescribing the transversals to any one-parameter family of curves C which simply cover the region R, that is, which cover the region in such a way that through each point there passes one and but one curve. In such a field of curves the direction angle τ is a function of x and y, and the transversal curves in the field are by definition the curves of a one-parameter family whose direction angles τ' satisfy the equation

$$(27) \qquad (f\cos\tau - f_\tau \sin\tau)\cos\tau' + (f\sin\tau + f_\tau\cos\tau)\sin\tau' = 0.†$$

If the transversals are prescribed as a one-parameter family of curves, $w(x, y) = \text{constant}$, simply covering the field and nowhere tangent to the original family, then τ' is given as a function of x and y by the equations

$$\cos\tau' = \frac{w_x}{\sqrt{w_x^2 + w_y^2}}, \qquad \sin\tau' = \frac{w_y}{\sqrt{w_x^2 + w_y^2}},$$

and is different at every point from $\tau(x, y)$. By substituting the value (25) in equation (27) it follows that the function θ must satisfy the differential equation

$$(28) \qquad \frac{\partial\theta}{\partial x}\cos\tau' + \frac{\partial\theta}{\partial y}\sin\tau' + Q(x, y) = 0,$$

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* Goursat-Hedrick, *A Course in Analysis*, vol. 1, §25.

† Bliss, A new form of the simplest problem of the calculus of variations, *Transactions Amer. Math. Soc.*, vol. 8 (1907), p. 413.

where

$$Q(x, y) = A_0 \cos\tau' + B_0 \sin\tau' + \int_0^{\tau} \sin(\tau' - \lambda) M(x, y, \lambda) d\lambda.$$

The most general solution of equation (28) can readily be determined if a particular solution $\chi(x, y)$ is known. For the function $w(x, y)$ is a solution of the abbreviated equation

$$\frac{\partial w}{\partial x} \cos\tau' + \frac{\partial w}{\partial y} \sin\tau' = 0, \tag{29}$$

and w and $\theta - \chi$ are two independent integrals of the auxiliary equation

$$\Phi_x \cos\tau' + \Phi_y \sin\tau' - Q\Phi_\theta = 0.$$

The most general solution of (28) is therefore determined by the equation

$$\theta - \chi = H(w),$$

where H is an arbitrary function of w.

Along any particular curve C_0 belonging to the original field, w varies monotonically, for on account of equation (29) it follows that

$$\frac{dw}{ds} = w_x \cos\tau + w_y \sin\tau \neq 0.$$

If this inequality were not true then at some point the ratios $\cos\tau : \sin\tau$ and $\cos\tau' : \sin\tau'$ would be equal and the transversal at that point would be tangent to C_0. Consequently s is also a monotonic function of w. If now the values of f as a function of s are given along C_0, then the arbitrary function H is determined except for an additive constant as a function of s by the equation

$$f(s) = \int_0^{\tau} \sin(\tau - \lambda) M(x, y, \lambda) d\lambda + (A_0 + \chi_x)\cos\tau + (B_0 + \chi_y)\sin\tau + \frac{dH}{ds},$$

and therefore, from what has just been said, as a function of w.

The results of this section are as follows:

If the boundary of the region R consists of a simply closed curve B which at no point satisfies the Euler equation, then the arbitrary function $G(u, v)$ is completely determined when the values which $M = f + f_{\tau\tau}$ takes at points of the boundary B for all values of τ, are assigned. Furthermore, the function $\theta(x, y)$ is determined, except for an additive constant, when, for any one-

parameter family of curves C which simply cover the region R, the family of transversal curves is assigned and the values of f along a particular curve C_0 are given. The function f is then uniquely determined in all that portion of the region R which is swept out by the transversal curves which intersect C_0.

4. The integrals having straight lines as extremals. As an application of the results of the preceding section consider the integrals which have as extremals the straight lines of the plane

$$x = t\cos v + u\sin v, \qquad y = t\sin v - u\cos v.$$

Here u is the distance of the line from the origin, and v is the angle which it makes with the positive x-axis. The solutions of these equations for t, u, v in terms of x, y, τ are

$$t = x\cos\tau + y\sin\tau, \quad u = x\sin\tau - y\cos\tau, \quad v = \tau,$$

and the differential equations (15) are simply

$$\frac{dx}{ds} = \cos\tau, \quad \frac{dy}{ds} = \sin\tau, \quad \frac{d\tau}{ds} = 0.$$

The value of $M(x, y, \tau)$ from equation (19) is

$$M = G(x\sin\tau - y\cos\tau, \tau),$$

the value $\psi = 1$ being an admissable value of ψ. Finally A_0 and B_0 can be taken equal to zero since the function $\phi(x, y, 0)$ in (24) vanishes. *The most general function f whose integral* (3) *has straight lines as extremals is therefore*

$$(30) \quad f = \int_0^\tau \sin(\tau - \lambda)\, G(x\sin\lambda - y\cos\lambda, \lambda)\,d\lambda + \theta_x\cos\tau + \theta_y\sin\tau,$$

where G is an arbitrary function of λ and $u = x \sin \lambda - y \cos \lambda$, and θ is an arbitrary function of x and y.

If the values of $M(x, y, \tau)$ at points of a circle

$$x^2 + y^2 = a^2$$

are assumed to be always equal to unity, then from the results of the last section $G(u, v)$ must be identically equal to unity for all values of u and v corresponding to straight lines which cut the circle. If the straight lines parallel to the x-axis are assumed to have the straight lines parallel to the

y-axis as transversals, then in equation (28) $\tau = 0$, $\tau' = \pi/2$, and the equation reduces to

$$\frac{\partial \theta}{\partial y} = 0.$$

Consequently θ is a function of x alone. If the values of f on the line $y = 0$ are assumed to be all equal to unity, then from (30)

$$1 = \theta_x, \quad \theta = x + c,$$

and it follows that the function f is everywhere equal to unity.

The integral

$$\int \sqrt{x_t^2 + y_t^2}\, dt$$

is therefore the most general integral (3) *for which*:

a) *straight lines are extremals,*

b) *the function $M(x, y, \tau)$ equals unity for values (x, y) on a circle with its center at the origin and for all values of τ,*

c) *the family of straight lines $y = c$ has as its family of transversals the lines $x = c$,*

d) *on the line $y = 0$ the values of $f(x, 0, 0)$ are unity.*

One of the problems which Darboux solved * was the determination of the form which the length integral on a surface has when the geodesics on the surface are represented in the plane of the parameters of the surface as straight lines. If the variables x and y are the parameters, the length integral will be an integral (3) in which

$$(31) \qquad f(x, y, \tau) = \sqrt{E \cos^2 \tau + 2F \cos \tau \sin \tau + G \sin^2 \tau},$$

E, F, and G being functions of x and y only. The problem is then to determine the functions f which are simultaneously of the forms (30) and (31). For (31) the value of M is

$$M = \frac{EG - F^2}{f^3},$$

and this must be a function of τ and $u = x \sin \tau - y \cos \tau$. The same must be true of $M^{-2/3}$, that is,

$$(32) \qquad \frac{E \cos^2 \tau + 2F \cos \tau \sin \tau + G \sin^2 \tau}{(EG - F^2)^{2/3}} = \Phi(u, \tau).$$

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* Loc. cit., §607.

The functions E, F, and G can be determined from this identity if $\Phi(u, \tau)$ is any homogeneous expression of the second degree in u, $\cos\tau$, $\sin\tau$. For since $u = x\sin\tau - y\cos\tau$, Φ may be arranged in powers of $\cos\tau$ and $\sin\tau$,

$$\Phi(u, \tau) = A\cos^2\tau + 2B\cos\tau\sin\tau + C\sin^2\tau, \tag{33}$$

and it follows that

$$E = \frac{A}{(AC - B^2)^2}, \quad F = \frac{B}{(AC - B^2)^2}, \quad G = \frac{C}{(AC - B^2)^2}.$$

But it is also true that Φ must be such a homogeneous expression in u, $\cos\tau$, $\sin\tau$. For by differentiating the left member of (32) for τ it follows that

$$\frac{d^3\Phi}{d\tau^3} + 4\frac{d\Phi}{d\tau} = 0,$$

and therefore

$$u_\tau^3\Phi_{uuu} + 3u_\tau^2\Phi_{uu\tau} + 3u_\tau(\Phi_{u\tau\tau} - u\Phi_{uu} + \Phi_u) + (\Phi_{\tau\tau\tau} - 3u\Phi_{u\tau} + 4\Phi_\tau) = 0.$$

This is an identity in x, y, τ and must also be an identity in τ, u, u_τ, since a set of values (x, y, τ) can be determined which gives τ, u, u_τ any arbitrarily assigned values. Hence by equating coefficients of powers of u_τ to zero, it can be readily seen that

$$\Phi = u^2K(\tau) + uL(\tau) + M(\tau),$$

where

$$\frac{dK}{d\tau} = 0, \quad \frac{d^2L}{d\tau^2} + L = 0, \quad \frac{d^3M}{d\tau^3} + 4M = 0,$$

from which it follows that K is a constant while L and M are any homogeneous expressions of the first and second degrees, respectively, in $\cos\tau$ and $\sin\tau$.

Hence the only possible length integrals for surfaces whose geodesics are to be represented by straight lines in the xy-plane, i. e., the plane of the parameters of the surface, are integrals of the form

$$f(x, y, \tau) = \frac{\sqrt{\Phi(u, \cos\tau, \sin\tau)}}{AC - B^2}.$$

In this expression Φ *may be any homogeneous polynomial of the second order in the three variables* $u = x\sin\tau - y\cos\tau$, $\cos\tau$, $\sin\tau$, *and* A, B, C *are the coefficients of its expansion* (33) *in powers of* $\cos\tau$ *and* $\sin\tau$.

PRINCETON UNIVERSITY,
PRINCETON, N. J.